International Journal of Theoretical Physics, Vol. 41, No. 7, July 2002 (© 2002)

Field Theory Methods in Classical Dynamics

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Received December 4, 2001

A Dirac picture perturbation theory is developed for the time evolution operator in classical dynamics in the spirit of the Schwinger–Feynman–Dyson perturbation expansion and detailed rules are derived for computations. Complexification formalisms are given for the time evolution operator suitable for phase space analyses, and then extended to a two-dimensional setting for a study of the geometrical Berry phase as an example. Finally a direct integration of Hamilton's equations is shown to lead naturally to a path integral expression, as a resolution of the identity, as applied to arbitrary functions of generalized coordinates and momenta.

KEY WORDS: quantum field theory; time evolution in classical dynamics; phase space complexification; Berry phase; path integrals.

1. INTRODUCTION

With the rapid progress of field theory in describing the basic interactions occurring in nature, it is becoming more and more important to extend the powerful techniques of field theory in describing physics, in general, in a more unified language. Such a unification has always been the goal of physics research over the years. The present work is aimed in describing classical dynamics in the spirit of field theory methods. Some earlier papers on this subject are given in Abrikosov (1993), Gozzi *et al.* (1989), Schwartz (1976), and Wetterich (1997), which are, however, only tangentially related to our present investigations. We develop a Dirac (interaction) picture perturbation theory in the spirit of the Schwinger–Feynman–Dyson perturbation theory to all orders in a coupling parameter and detailed *rules* are derived for computations starting from the time evolution of any function of the generalized coordinates and momenta. Complexifications of the time evolution are developed suitable for phase space analyses and then extended to a two-dimensional setting to describe the so-called geometrical Berry phase (Berry, 1984; Shapere and Wilczek, 1989) as an example. Finally, since the classical limit of the path integral, starting from the quantum regime, reduces to just a phase factor

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0020-7748/02/0700-1327/0

involving the classical action, such a limit (Abrikosov, 1993; Gozzi *et al.*, 1989) is not of very practical value, as it stands, for actual computations. Instead, we develop a path integral expression by direct integration of Hamilton's equations, as a resolution of the identity, that may be applied to any function of the generalized coordinates and momenta, in the same spirit of developing the resolution of the identity of a self-adjoint operator, in quantum physics, that may be applied to any vector in the underlying Hilbert space. The perturbation expansion, with the derived rules, is given in Section 2, while the complexification formalisms are developed in Section 3. Section 4 deals with the path integral expression and its consistency with the Poisson-bracket solution.

2. RULES FOR COMPUTATIONS AND THE DIRAC PICTURE PERTURBATION THEORY

The time derivative of an arbitrary function $f[q(t), p(t)]$ of generalized coordinates and generalized momenta, via the Poisson-bracket formalism, reads

$$\frac{d}{dt} f[q(t), p(t)] = \left[\frac{\partial H(t)}{\partial p(t)}\frac{\partial}{\partial q(t)} - \frac{\partial H(t)}{\partial q(t)}\frac{\partial}{\partial p(t)}\right] f[q(t), p(t)] \equiv f_1[q(t), p(t)] \tag{2.1}$$

where $H[q(t), p(t)] \equiv H(t)$ is the Hamiltonian, constructed out of the variables $q(t)$ and $p(t)$, assumed with no explicit time dependence. Similarly,

$$\left(\frac{d}{dt}\right)^n f[q(t), p(t)] \equiv f_n[q(t), p(t)] = \left[\frac{\partial H(t)}{\partial p(t)}\frac{\partial}{\partial q(t)} - \frac{\partial H(t)}{\partial q(t)}\frac{\partial}{\partial p(t)}\right]^n f[q(t), p(t)] \tag{2.2}$$

leading to the familiar time evolution

$$f[q(t), p(t)] = \exp[tO]\, f[q, p] \tag{2.3}$$

where

$$O = \left[\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}\right] \tag{2.4}$$

and $q = q(0)$, $p = p(0)$, $H = H(0)$.

Upon using the integral identity

$$e^{t[A+\lambda B]}\, e^{-tA} = 1 + \lambda \int_0^t dt_1\, e^{t_1[A+\lambda B]} B\, e^{-t_1 A} \tag{2.5}$$

for two operators A and B, and setting

$$H = H_1 + \lambda H_2 \tag{2.6}$$

$$A = \left[\frac{\partial H_1}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H_1}{\partial q} \frac{\partial}{\partial p} \right] \tag{2.7}$$

$$B = \left[\frac{\partial H_2}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H_2}{\partial q} \frac{\partial}{\partial p} \right] \tag{2.8}$$

one obtains, upon iteration of (2.5), the expression

$$f[q(t), p(t)] = \sum_{n=0}^{\infty} (\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n B(t_n) \cdots B(t_1)\, e^{tA} f[q, p] \tag{2.9}$$

where

$$B(t) = e^{tA} B\, e^{-tA} \tag{2.10}$$

with $B(t)$ independent of the parameter λ.

In particular, for

$$H_1 = p^2/2m, \qquad H_2 = V(q) \tag{2.11}$$

$$B(t) = F\left[q + \frac{p}{m}t\right] \left[\frac{\partial}{\partial p} - \frac{t}{m} \frac{\partial}{\partial q} \right] \tag{2.12}$$

$$F[q] = -V'(q) \tag{2.13}$$

$$f(q, p) = q \tag{2.14}$$

then

$$e^{tA} f(q, p) = q + \frac{t}{m} p \tag{2.15}$$

and we obtain

$$q(t) = \sum_{n=0}^{\infty} (\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\ B(t_n) \cdots B(t_1) \left[q + \frac{t}{m} p \right] \tag{2.16}$$

In detail

$$B(t_1)\left[q + \frac{t}{m}p\right] = \left(\frac{t - t_1}{m}\right) F \left[q + \frac{t_1}{m} p\right] \tag{2.17}$$

$$B(t_2)B(t_1)\left[q + \frac{t}{m}p\right] = \left(\frac{t_1 - t_2}{m}\right)\left(\frac{t - t_1}{m}\right) F\left[q + \frac{t_2}{m}p\right] F'\left[q + \frac{t_1}{m}p\right] \tag{2.18}$$

and so on. Upon letting

$$q + \frac{t_i}{m} p = u_i \tag{2.19}$$

we obtain, after a systematic analysis, the following general explicit rule:

$$B(t_n)\cdots B(t_1)\left[q+\frac{t}{m}p\right]=\left(\frac{t-t_1}{m}\right)F[u_n]\sum\left(\frac{t_1-t_2}{m}\right)^{\delta(k_1,2)}\cdots$$
$$\times\left(\frac{t_1-t_n}{m}\right)^{\delta(k_1,n)}F^{(k_1)}[u_1]$$
$$\times\left(\frac{t_2-t_3}{m}\right)^{\delta(k_2,3)}\cdots\left(\frac{t_2-t_n}{m}\right)^{\delta(k_2,n)}$$
$$\times F^{(k_2)}[u_2]\cdots\left(\frac{t_{n-1}-t_n}{m}\right)^{\delta(k_{n-1},n)}$$
$$\times F^{(k_{n-1})}[u_{n-1}] \tag{2.20}$$

where

$$F^{(a)}[u]=\left(\frac{d}{du}\right)^a F[u] \tag{2.21}$$

and the sum in (2.20) is over all k's and δ's such that

$$\left.\begin{array}{l} k_1+k_2+\cdots+k_{n-1}=n-1\\ k_1=1,\ldots,n-1\\ k_2=0,1,\ldots,n-2\\ \vdots\\ k_{n-1}=0,1\end{array}\right\} \tag{2.22}$$

and

$$\delta(k_{i,j})=0,\quad \text{if}\quad k_i=0 \tag{2.23}$$

$$\delta(k_{i,j})=0,\quad \text{if}\quad 1\le j\le i \tag{2.24}$$

and for $j>i$, the $\delta(k_{i,j})$ are zero or one such that

$$\sum_{j=i+1}^{n}\delta(k_{i,j})=k_i,\quad i=1,\ldots,n-1 \tag{2.25}$$

and

$$\sum_{i=1}^{n-1}\delta(k_{i,j})=1,\quad (\delta(k_{i,j})=0,\quad j=1,2,\ldots,i) \tag{2.26}$$

that is, for a fixed j, t_j appears only once in the product:

$$\prod_{j=2}^{n}(t_1-t_j)^{\delta(k_{1,j})}\prod_{j=3}^{n}(t_2-t_i)^{\delta(k_{2,j})}\cdots \tag{2.27}$$

For example, for $n = 4$,

$$k_1 = 1, 2, 3; \quad k_2 = 0, 1, 2; \quad k_3 = 0, 1; \quad k_1 + k_2 + k_3 = 3 \tag{2.28}$$

and for $k_1 = 2, k_2 = 1, k_3 = 0$,

$$\delta(2, 2) + \delta(2, 3) + \delta(2, 4) = 2 \tag{2.29}$$

$$\delta(1, 3) + \delta(1, 4) = 1 \tag{2.30}$$

and

$$\delta(2, j) + \delta(i, j) = 1 \tag{2.31}$$

For the harmonic oscillator $V(q) = q^2/2$, $\lambda = m\omega^2$, and

$$F[q] = -q, \qquad B = -q\frac{\partial}{\partial p}, \qquad A = \frac{p}{m}\frac{\partial}{\partial q} \tag{2.32}$$

The only solution being

$$k_1 = \cdots = k_{n-1} = 1 \tag{2.33}$$

with

$$F^{(1)}[u_i] = -1, \quad i = 1, \ldots, n-1; \qquad F[u_n] = -[q + t_n p/m] \tag{2.34}$$

we obtain

$$q(t) = \sum_{n=0}^{\infty}(-\omega^2)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n (t - t_1) \times (t_1 - t_2)\cdots(t_{n-1} - t_n)\left[q + \frac{t_n}{m}p\right] \tag{2.35}$$

which integrates out to

$$q(t) = q\sum_{n=0}^{\infty}(-\omega^2)^n \frac{(t)^{2n}}{(2n)!} + \frac{p}{m}\sum_{n=0}^{\infty}(-\omega^2)^n \frac{(t)^{2n+1}}{(2n+1)!} \tag{2.36}$$

or

$$q(t) = q\cos\omega t + \frac{p}{m}\frac{\sin\omega t}{\omega}. \tag{2.37}$$

3. COMPLEXIFICATION OF THE TIME EVOLUTION

We consider the complex dynamical variable

$$Z(t) = aq(t) + ibp(t) \tag{3.1}$$

where a and b are arbitrary real constants. Equation (2.9) then immediately leads to

$$Z(t) = \sum_{n=0}^{\infty} (\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \; B(t_n)B(t_{n-1}) \cdots$$
$$\times B(t_1) \left[a \left(q + \frac{t}{m} p \right) + ibp \right] \tag{3.2}$$

in which $B(t_n) \cdots B(t_1)[a(q + tp/m) + ibp]$ coincides with the expression in (2.20) except that the overall factor $(t - t_1)/m$ in the latter is replaced by

$$\left[a \left(\frac{t - t_1}{m} \right) + ib \right] \tag{3.3}$$

A more explicit expression may be also obtained directly from (2.3). To this end, consider the Hamiltonian

$$H = \frac{p^2}{2m} + m\omega^2 \frac{q^2}{2} + \lambda V(q) \tag{3.4}$$

and upon setting

$$Z(t) = q(t) + ip(t)/m\omega \tag{3.5}$$

one readily obtains

$$Z(t) = \exp\left[-i \frac{t\omega}{2} \hat{C} \right] Z(0) \tag{3.6}$$

where

$$\hat{C} = (Z - Z^*) \left(\frac{\partial}{\partial Z} + \frac{\partial}{\partial Z^*} \right) + \left((Z + Z^*) - 2\lambda \frac{F[(Z + Z^*)/2]}{m\omega^2} \right)$$
$$\times \left(\frac{\partial}{\partial Z} - \frac{\partial}{\partial Z^*} \right) \tag{3.7}$$

where $F[\cdot]$ is defined in (2.13). For $V(q) = -q$,

$$\hat{C} = \left[2 \left(Z - \frac{\lambda}{m\omega^2} \right) \frac{\partial}{\partial Z} - 2 \left(Z^* - \frac{\lambda}{m\omega^2} \right) \frac{\partial}{\partial Z^*} \right] \tag{3.8}$$

leading from

$$Z(t) = \exp\left[-it\omega \left(Z - \frac{\lambda}{m\omega^2} \right) \frac{\partial}{\partial Z} \right] Z \tag{3.9}$$

and upon using the identity

$$\exp\left(aZ \frac{\partial}{\partial Z} \right) Z = Z \exp a \tag{3.10}$$

for any constant a, to the expression

$$Z(t) = \frac{\lambda}{m\omega^2} + \exp(-it\omega)\left(Z - \frac{\lambda}{m\omega^2}\right). \tag{3.11}$$

A more interesting application is to the geometrical phase associated with the famous Foucault pendulum with Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{mg}{2L}(x^2 + y^2) + (p_x y - p_y x)\omega_z \tag{3.12}$$

where $\sqrt{g/L} \equiv \omega_0 \gg \omega$, $\omega_z = \omega\ \sin\lambda$, with λ denoting the latitude and L the length of the pendulum. Then

$$\begin{aligned} O = {} & \left(\frac{p_x}{m} + y\omega_z\right)\frac{\partial}{\partial x} - \left(m\omega_0^2 x - p_y\omega_z\right)\frac{\partial}{\partial p_x} \\ & + \left(\frac{p_y}{m} - x\omega_z\right)\frac{\partial}{\partial y} - \left(m\omega_0^2 y + p_x\omega_z\right)\frac{\partial}{\partial p_y} \end{aligned} \tag{3.13}$$

Upon defining

$$U = \left(x + \frac{ip_x}{m}\right) + i\left(y + \frac{ip_y}{m}\right) \tag{3.14}$$

$$V = \left(x + \frac{ip_x}{m}\right) - i\left(y + \frac{ip_y}{m}\right) \tag{3.15}$$

then

$$Z = x + iy = \frac{U + V^*}{2} \tag{3.16}$$

and

$$\begin{aligned} O = {} & \left(\frac{\omega_0 + \omega_z}{i}\right)U\frac{\partial}{\partial U} - \left(\frac{\omega_0 + \omega_z}{i}\right)U^*\frac{\partial}{\partial U^*} \\ & + \left(\frac{\omega_0 - \omega_z}{i}\right)V\frac{\partial}{\partial V} - \left(\frac{\omega_0 - \omega_z}{i}\right)V^*\frac{\partial}{\partial V^*} \end{aligned} \tag{3.17}$$

which lead to

$$Z(t) = \exp\left(-it(\omega_0 + \omega_z)U\frac{\partial}{\partial U}\right)\ \exp\left(it(\omega_0 - \omega_z)V^*\frac{\partial}{\partial V^*}\right)\left(\frac{U + V^*}{2}\right) \tag{3.18}$$

With initial conditions

$$x(0) = e^{i2\pi}x_0, \quad y(0) = 0, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 0 \tag{3.19}$$

where $e^{i2\pi}$ denotes the initial phase of the plane of oscillations of the pendulum moving clockwise, we have

$$U(0) = e^{i2\pi} x_0 = V^*(0) \tag{3.20}$$

giving

$$Z(t) = \exp(-it\omega_z)\, e^{i2\pi} x_0 \tag{3.21}$$

For $t = 2\pi/\omega$, corresponding to a full rotation of the earth,

$$Z\left(\frac{2\pi}{\omega}\right) = \exp(i2\pi(1 - \sin\lambda))x_0 \cos\omega_0 t \tag{3.22}$$

giving the familiar geometrical Berry phase $\exp i2\pi(1 - \sin\lambda)$, which is independent of g and ω, with $2\pi(1 - \sin\lambda)$ denoting the solid angle subtended at the center of the earth.

4. A PATH INTEGRAL EXPRESSION

Upon integrating Hamilton's equations $\partial H/\partial p = \dot{q}$, $\partial H/\partial q = -\dot{p}$ we have

$$\begin{aligned} f[q(t), p(t)] = f\Bigg[&q(0) + \int_0^t dt'\partial H(t')/\partial p(t'), \\ &p(0) - \int_0^t dt'\partial H(t')/\partial q(t')\Bigg] \end{aligned} \tag{4.1}$$

We devide the time interval from 0 to t into n subintervals each of length t/n. Evaluating each of the summands at the left-end points of these subintervals, we may rewrite the right-hand side of (4.1) as

$$\lim_{n\to\infty} f\left[q_0 + \frac{t}{n}\sum_{k=0}^{n-1}\partial H_k/\partial p_k, \qquad p_0 - \frac{t}{n}\sum_{k=0}^{n-1}\partial H_k/\partial q_k\right] \tag{4.2}$$

where now we set $q(0) = q_0$, $p(0) = p_0$, and k is a short-hand for tk/n. The expression (4.2) to which the limit $n \to \infty$ is to be taken may be equivalently rewritten as

$$\begin{aligned} &\int dq_n dp_n \delta\left(q_0 + \frac{t}{n}\sum_{k=0}^{n-1}\frac{\partial}{\partial p_k}H_k,\, p_0 - \frac{t}{n}\sum_{k=0}^{n-1}\frac{\partial H_k}{\partial q_k}\right) f[q_n, p_n] \\ &= \int\left(\prod_{k=1}^{n} dq_k\, dp_k\, \delta\left(q_{k-1} + \frac{t}{n}\frac{\partial}{\partial p_{k-1}}H_{k-1} - q_k\right)\right. \\ &\quad \left.\times\, \delta\left(p_{k-1} - \frac{t}{n}\frac{\partial}{\partial q_{k-1}}H_{k-1} - p_k\right)\right) f[q_n, p_n] \end{aligned}$$

$$= \int \left(\prod_{k=1}^{n} dq_k \, dp_k \, \delta \left(\frac{t}{n} \left(\frac{\partial}{\partial p_{k-1}} H_{k-1} - \frac{(q_k - q_{k-1})}{t/n} \right) \right) \right.$$

$$\left. \times \, \delta \left(\frac{t}{n} \left(\frac{\partial}{\partial q_{k-1}} H_{k-1} + \frac{(p_k - p_{k-1})}{t/n} \right) \right) \right) f[q_n, p_n] \tag{4.3}$$

Upon taking the limit $n \to \infty$ of (4.3), Eq. (4.1) becomes

$$f[q(t), p(t)] = \int [dq][dp] \delta \left(\frac{\partial H}{\partial p} - \dot{q} \right) \delta \left(\frac{\partial H}{\partial q} + \dot{p} \right) f[q(t), p(t)] \tag{4.4}$$

as a resolution of the identity as applied to arbitrary functions $f[q(t), p(t)]$ of $q(t)$ and $p(t)$. The integrals are over all paths in phase space, from 0 to t, starting from $(q(0), p(0))$, and the delta functionals as obtained directly from Hamilton's equations restrict these paths to the classic one obeying Hamilton's equations at each instant of time. Finally, we have also used the fact that $q_n \to q(t)$, $p_n \to p(t)$ from the very definitions in (4.2), (4.3) for $n \to \infty$. Equation (4.4) is in the spirit of the resolution of the identity of a self-adjoint operator in quantum physics as applied to any arbitrary vector in the underlying Hilbert space.

For applications, (4.4) may be rewritten in the more manageable form

$$\int [dq][dp][d\lambda][d\eta] \, \exp i \int_0^t dt' \left[\lambda(t') \left(\frac{\partial H(t')}{\partial p(t')} - \dot{q}(t') \right) \right.$$

$$\left. + \, \eta(t') \left(\frac{\partial H(t')}{\partial q(t')} + \dot{p}(t') \right) \right] f[q(t), p(t)] \tag{4.5}$$

in terms an uncountable infinite number of Lagrange multipliers $\lambda(\cdot)$, $\eta(\cdot)$.

For example, for a charged particle of charge e in a uniform magnetic field B, say, along the z-axis, we may write for the vector potential $\vec{A} = (-q_2, q_1)B/2$, with motion in a plane, and for the Hamiltonian H

$$2mH = (p_1 + q_2 m\omega/2)^2 + (p_2 - q_1 m\omega/2)^2 \tag{4.6}$$

with $\omega \equiv eB/mc$, $\lambda = (\lambda_1, \lambda_2)$, $\eta = (\eta_1, \eta_2)$. The time-integrand in the exponential in (4.5), without the i factor, may be rewritten as

$$\frac{\lambda_1}{m} \left[\left(p_1 + q_2 \frac{m\omega}{2} \right) - \dot{q}_1 m \right] + \eta_1 \left[- \left(p_2 - q_1 \frac{m\omega}{2} \right) \frac{\omega}{2} + \dot{p}_1 \right]$$

$$+ \, \frac{\lambda_2}{m} \left[\left(p_2 - q_1 \frac{m\omega}{2} \right) - \dot{q}_2 m \right] + \eta_2 \left[\left(p_1 + q_2 \frac{m\omega}{2} \right) \frac{\omega}{2} + \dot{p}_2 \right]$$

$$
\begin{aligned}
&\equiv (-\lambda_1/2m\omega)[(d/dt' + i\omega)U + (d/dt' - i\omega)U^* + d(V + V^*)/dt'] \\
&\quad + (\eta_1/4)\left[\frac{1}{i}(d/dt' + i\omega)U - \frac{1}{i}(d/dt' - i\omega)U^* + \frac{1}{i}d(V - V^*)/dt'\right] \\
&\quad - (\lambda_2/2m\omega)\left[\frac{1}{i}(d/dt' + i\omega)U - \frac{1}{i}(d/dt' - i\omega)U^* - \frac{1}{i}d(V - V^*)/dt'\right] \\
&\quad - (\eta_2/4)[(d/dt' + i\omega)U + (d/dt' - i\omega)U^* - d(V + V^*)/dt'] \qquad (4.7)
\end{aligned}
$$

where

$$U = (q_1 m\omega/2 - p_2) + i(p_1 + q_2 m\omega/2) \qquad (4.8)$$

$$V = (q_1 m\omega/2 + p_2) + i(p_1 - q_2 m\omega/2) \qquad (4.9)$$

The coefficients of λ_1, η_1, λ_2, η_2 on the right-hand sides of (4.7) are all reals. Upon integration over λ_1, η_1, λ_2, η_2, we learn that the real and imaginary parts of

$$\left(\frac{d}{dt} + i\omega\right)U \pm \frac{dV}{dt}$$

must vanish. That is, $V(t) = V(0)$, and $(d/dt + i\omega)U = 0$ or $U(t) = U(0)\exp - i\omega t$. Upon equating the real and imaginary parts of each of the latter two equations we obtain the solution

$$
\begin{aligned}
q_1(t) &= (q_1(0)/2 + p_2(0)/m\omega) + (q_1(0)/2 - p_2(0)/m\omega)\ \cos\omega t \\
&\quad + (p_1(0)/m\omega + q_2(0)/2)\ \sin\omega t \qquad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
q_2(t) &= (q_2(0)/2 - p_1(0)/m\omega) + (q_2(0)/2 + p_1(0)/m\omega)\ \cos\omega t \\
&\quad + (-q_1(0)/2 + p_2(0)/m\omega)\ \sin\omega t \qquad (4.11)
\end{aligned}
$$

To check the consistency of (4.4), we note that in reference to the first equality on the right-hand side of (4.3), that

$$\exp\frac{t}{n}\left(\frac{\partial H_{k-1}}{\partial p_{k-1}}\right)\frac{\partial}{\partial q_{k-1}}$$

is not quite a translation operator for a function of q_{k-1}, since $\partial H_{k-1}/\partial p_{k-1}$ may, in general, depend on q_{k-1} as well. However, in view of the fact that the limit $n \to \infty$ is to be taken, this operator may be indeed taken to have such a property for the accuracy needed. A similar comment applies to the

$$\exp -\frac{t}{n}\left(\frac{\partial H_{k-1}}{\partial q_{k-1}}\right)\frac{\partial}{\partial p_{k-1}}$$

operator. To the accuracy needed (4.1)–(4.3) lead to

$$f[q(t),\, p(t)] = \lim_{n\to\infty} \int \left(\prod_{k=1}^{n} dq_k dp_k \left[e^{\frac{t}{n}(\partial H_{k-1}/\partial p_{k-1})\partial/\partial q_{k-1}}\; e^{-\frac{t}{n}(\partial H_{k-1}/\partial q_{k-1})\partial/\partial p_{k-1}} \right.\right.$$

$$\left.\left. \times\ \delta(q_{k-1} - q_k)\delta(p_{k-1} - p_k) \right] \right) f[q_n,\, p_n] \qquad (4.12)$$

Upon integration over $(q_k,\, p_k)$ with the aid of the delta functions which eventually pick up the $(q_0,\, p_0)$ value for the former, we obtain the Poisson-bracket solution

$$\begin{aligned} &f[q(t),\, p(t)] \\ &\quad = \lim_{n\to\infty} \left(\exp \frac{t}{n} \frac{\partial}{\partial p(0)} H(0) \frac{\partial}{\partial q_0}\ \ \exp -\frac{t}{n} \frac{\partial}{\partial q(0)} H(0) \frac{\partial}{\partial p_0} \right)^n f[q_0,\, p_0] \\ &\quad = \exp \left(t \left[\frac{\partial}{\partial p(0)} H(0) \frac{\partial}{\partial q_0} - \frac{\partial H(0)}{\partial q(0)} \frac{\partial}{\partial p_0} \right] \right) f[q_0,\, p_0]. \end{aligned} \qquad (4.13)$$

ACKNOWLEDGMENTS

This work was supported by a Royal Golden Jubilee Award.

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